ON THE EQUATIONS OF MOTION FOR SYSTEMS WITH NONIDEAL CONSTRAINTS

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The mechanics of systems subject to constraints is based on the assumption that the equations of constraints are satisfied exactly during the whole course of motion for arbitrary forces acting on the system and for arbitrary initial conditions consistent with the constraints. It is clear, however, that in an arbitrary mechanical system, the model of which is a system with constraints (for example, ideal and holonomic), the reactions of constraints arise in consequence of violating the latter. In the majority of problems known in the literature the reactions arise in consequence of elastic deformations of bodies belonging to the system and those exterior to the latter.

Since the disturbances of the reactions which arise during the motion are very small, their neglect does not cause any essential differences between the theory and experiment. This model together with a certain hypothesis concerning the properties of reactions (their being ideal) permits us to obtain in quite a general form the equations of motion for systems with ideal holonomic as well as nonholonomic constraints, and also the equations of motion for holonomic systems with friction [1], provided that a certain law of friction holds.

However, due to a frequent appearance of automatically controlled systems, it seems to us interesting to consider systems with automatic devices, the action of which can be interpreted as that of a constraint, not belonging to any of the above-mentioned types.

Beghin [2] considered first constraints of a similar type, with certain partial restrictions on their reactions.

1. Let us begin our consideration with some examples.

(a) Let, at the end A of a flexible and inextensible string which can glide through a small ring, be suspended a heavy material point of mass m. Let an automatic device P, to which the other end is connected, realize a constraint in the form of a relation $r = r(a)$ between the angle of deviation a of the string from the vertical and the distance $OA = r$.

Then the angular-momentum theorem with respect to the point O gives

$$\frac{d}{dt}(mr^2\dot{\alpha}) = -mgr\sin\alpha \tag{1.1}$$

and the reaction N is determined by the relation $-N + mg\cos a = m(r - a^2r)$. After substituting for a its expression from the first equation, and taking into account the equation of the constraint, N becomes a single-valued function of a and a.

(b) A smooth rod, which can rotate about a point O, carries a heavy material point A which can glide without friction along the rod. An automatic device P acting on the rod realizes a constraint $r = r(a)$. The reaction N of the rod is perpendicular to the rod. The equation of motion in the projection on the rod assumes the form

$$m(\ddot{r} - r\dot{\alpha}^2) = mg\cos\alpha \tag{1.2}$$

and the reaction N is determined by the relation

$$(N - mg\sin\alpha)\,r = \frac{d}{dt}(mr^2\dot{\alpha})$$

Using the preceding equation and the equation of the constraint, N can be determined as a function of a and $\dot{a}$.

In both of the above-mentioned cases the system will actually move along the trajectory $r(a) + \delta r(a)$, where $\delta r(a)$ denotes a small deviation from the equation of the constraint, which by way of information enters the automatic device, and in consequence of which the reaction arises. However, if the system P is sufficiently sensitive, i.e. $\delta r(a)$ is comparable to the deviations arising in consequence of deformations of ideal holonomic constraints, then a similar model, apparently, deserves attention.

Let us note that in both examples, the equation of constraint and the acting forces being the same, the motion will be completely different because in the first example the reaction is directed along AO and in the second example perpendicular to AO. Both examples can be formally attributed to "systems with friction" according to Painlevé. However, it seems to us that in the present case the use of this term is not justified.

Let us also note that in both examples the automatic device can realize some other constraint $f(r, \dot{r}, a, \dot{a}) = 0$. Then, the first equation, not containing the reaction, together with the equation of the constraint, will represent a system of equations of motion, the form of the first of which, until the equation of the constraint is accounted for, does not depend on the form of the latter equation. This is easy to understand if one takes into account the fact that the first equation reflects specific circumstances of the given device. In the first example the reaction is directed along the string while in the second example the reaction is perpendicular to the rod.

2. Consider a system of material particles subject to ideal holonomic constraints. Let the position in space of this system be determined by holonomic coordinates $q_1, \ldots\ q_n$, and let it be acted on by generalized forces $Q_1, \ldots, Q_n$.

If, in addition, we impose on the system under consideration a certain number of ideal holonomic, linear nonholonomic or nonlinear nonholonomic constraints of the Chetaev type [3]

$$f_i(q_1, \ldots, q_n, t) = 0, \quad \sum_{j=1}^{n} A_{kj}\dot{q}_j + B_k = 0, \quad \psi_s(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t) = 0 \tag{2.1}$$

$$(i = 1, \ldots, r) \qquad (k = 1, \ldots, l) \qquad (s = 1, \ldots, p)$$

$$(r + l + p = m < n)$$

then the principle of virtual displacements

$$\sum_{i=1}^{n} \left[\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - Q_j\right]\delta q_j = 0$$

defining the latter by

$$\sum_{j=1}^{n} \frac{\partial f_i}{\partial q_j}\delta q_j = 0, \quad \sum_{j=1}^{n} A_{kj}\delta q_j = 0, \quad \sum_{j=1}^{n} \frac{\partial \psi_s}{\partial \dot{q}_j}\delta q_j = 0 \tag{2.2}$$

$$(i = 1, \ldots, r) \qquad (k = 1, \ldots, l) \qquad (s = 1, \ldots, p)$$

gives us the equations of motion.

The last equations can briefly be written in the form

$$a_{i1}\delta q_1 + \ldots + a_{in}\delta q_n = 0 \qquad (i = 1, \ldots, m)$$

and the equations of motion can be obtained if, and only if, the matrix (a_{ij}) is of rank m.

If the forces of reaction $R_1, \ldots, R_n$ are added to the forces acting on the system, then the system can be considered as unconstrained. The equation $R_1\delta q_1 + \ldots + R_n\delta q_n = 0$, together with Equations (2.2), then constitute a system of $m+1$ equations in $\delta q_1, \ldots, \delta q_n$, the rank of the corresponding matrix being m. Consequently, $R_1, \ldots, R_n$ satisfy $n-m$ linear equations which are obtained by equating to zero the $n-m$ minors of order $m+1$ of the matrix

$$\begin{Vmatrix} R_1 & \ldots & R_n \\ a_{11} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots \\ a_{m1} & \ldots & a_{mn} \end{Vmatrix}$$

Let these equations have the form

$$b_{i1}R_1 + \ldots + b_{in}R_n = 0 \qquad (i = 1, \ldots, n-m) \tag{2.3}$$

They determine the "directions" of the reactions in the space of q_i, $\dot{q}_i$, t.

Differentiate twice with respect to the time the first r equations of constraints and once the remaining ones. Next, replace the $\ddot{q}_j$ by the expressions obtained by solving the equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j + R_j \qquad (j = 1, \ldots, n)$$

for the $\ddot{q}_j$. Then equations

$$\sum_{j=1}^{n} c_{ij}(Q_j + R_j) = d_i \qquad (i = 1, \ldots, m) \tag{2.4}$$

are obtained, where c_{ij}, d_i are functions of q_j, $\dot{q}_j$, t. Equations (2.3) together with (2.4) determine the reactions as functions of $q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, Q_i, t$.

Let us notice that Equations (2.3) are obtained from the equations of constraints as a result of a certain operation. They are linear with respect to the reactions and do not contain the forces acting on the system.

3. Suppose now that the constraints (2.1) are not ideal, i.e. the reactions are not perpendicular to the quantities determined by (2.2). Assume, however, that they possess the property that in every admissible state $q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t$, i.e. consistent with the constraints, $\ddot{q}_1, \ldots, \ddot{q}_n$ are uniquely determined by the state of the system and the active forces $Q_1, \ldots, Q_n$. It can be shown that this assumption is equivalent to the supposition that $R_1, \ldots, R_n$ are uniquely determined by the admissible state of the system and the forces acting on it.

This means that the reactions, the admissible state of the system and the forces acting on it must be subject to $n-m$ relations

$$\Psi_i(R_1, \ldots, R_n, q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t, Q_1, \ldots, Q_n) = 0 \qquad (i = 1, \ldots, n-m) \tag{3.1}$$

which are such that (3.1) together with (2.4) determine the R_j in terms of the state of the system and the forces acting on it. The relations (3.1) have to be determined empirically.

These equations will be said to express the "axiom of reactions" for nonideal constraints (2.1).

If only the motion of the system is to be determined, and we are not interested in the determination of the reactions, then equations

$$\Psi_i\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1} - \frac{\partial T}{\partial q_1} - Q_1, \ldots, \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_n} - \frac{\partial T}{\partial q_n} - Q_n, q_j, \dot{q}_j, t, Q_j\right) = 0$$
$$(i = 1, \ldots, m) \tag{3.2}$$

together with those of the constraints (2.1) constitute the system of equations sought and do not contain the reactions.

If the axiom of reactions does not contain the active forces and the equations of constraints (2.1) are holonomic, it can be proved that the above set of hypotheses is equivalent to the set of hypotheses considered by Painlevé.

4. Consider a mechanical system with ideal constraints, the equations of which have the form (2.1), and the axiom of reactions

$$b_{i1}^\circ(q_1, \ldots, q_n, t) R_1 + \ldots + b_{in}^\circ(q_1, \ldots, q_n, t) R_n = 0 \quad (i = 1, \ldots, n-m) \tag{4.1}$$

is linear with respect to the R_j, the coefficients b_{ij}° depending only on the coordinates and the time. Suppose also that the system of equations

$$\overset{\circ}{a}_{k1}\delta q_1 + \ldots + \overset{\circ}{a}_{kn}\delta q_n = 0 \qquad (k = 1, \ldots, m) \tag{4.2}$$

obtained by equating to zero the minors of order $n - m + 1$ of the matrix

$$\left\| \begin{matrix} \delta q_1 & \ldots & \delta q_n \\ \overset{\circ}{b}_{11} & \ldots & \overset{\circ}{b}_{1n} \\ \cdot & \cdots & \cdot \\ b_{n-m1} & \ldots & b_{n-m,n} \end{matrix} \right\|$$

is a completely integrable Pfaffian system, the independent integrals of which are $y_1(q_1, \ldots, q_n, t), \ldots, y_m(q_1, \ldots, q_n, t)$.

As long as any row $\delta q_1, \ldots, \delta q_n$, satisfying Equations (4.2), possesses the property that $R_1\delta q_1 + \ldots + R_n\delta q_n = 0$, it also has the property

$$\sum_{j=1}^{n} \left[\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - Q_j\right] \delta q_j = 0 \tag{4.3}$$

Thus, Equations (4.2) can be considered as giving the definition of "virtual displacements" corresponding to the axiom of reactions (4.1).

Let $q_1^\circ, \ldots, q_n^\circ$ denote new generalized coordinates which are such that $q_1, \ldots, q_n$ can be expressed in terms of them in the form of independent, twice-continuously differentiable functions

$$q_i = \varphi_i(q_1^\circ, \ldots, q_n^\circ, t) \qquad (i = 1, \ldots, n) \tag{4.4}$$

In addition, after expressing the q_i in terms of the q_i° by means of Formulas (4.4), let $y_1, \ldots, y_m$ go over into $y_1^\circ(q_{n-m+1}^\circ, \ldots, q_n^\circ, t)$, $\ldots, y_m^\circ(q_{n-m+1}, \ldots, q_n^\circ, t)$ depending only on the last m new coordinates and the time.

It is easy to see that system (4.2) is equivalent to the system

$$\delta y_1 = \ldots = \delta y_m = 0$$

together with the system

$$\delta \overset{\circ}{q}_{n-m+1} = \ldots = \delta \overset{\circ}{q}_n = 0 \tag{4.5}$$

If now $T^\circ = T(q_1^\circ, \ldots, q_n^\circ, \dot{q}_1^\circ, \ldots, \dot{q}_n^\circ, t)$ and the Q_j° are the generalized forces corresponding to the new coordinates, then by virtue of (4.5) Equation (4.3) assumes the form

$$\sum_{j=1}^{n-m} \left[\frac{d}{dt}\frac{\partial T^\circ}{\partial \dot{q}_j} - \frac{nT^\circ}{\partial q_i^\circ} - Q_j^\circ\right] \delta q_j^\circ = 0$$

Since $\delta q_1^\circ, \ldots, \delta q_{n-m}^\circ$ represent arbitrary quantities, it follows necessarily from the last equation that

$$\frac{d}{dt}\frac{\partial T^\circ}{\partial \dot{q}_j^\circ} - \frac{\partial T}{\partial q_j^\circ} = Q_j^\circ \qquad (j = 1, \ldots, n-m) \tag{4.6}$$

Adjoining to these equations those of the constraints expressed in terms of the new coordinates

$$f_i^\circ(q_1^\circ, \ldots, q_n^\circ, t) = 0, \quad \sum \overset{\circ}{A}_{kj}\dot{\overset{\circ}{q}}_j + \overset{\circ}{B}_k = 0, \quad \psi_s(\overset{\circ}{q}_1, \ldots, \overset{\circ}{q}_n, \dot{\overset{\circ}{q}}_1, \ldots, \dot{\overset{\circ}{q}}_n, t) = 0$$
$$(i = 1, \ldots, r) \qquad (k = 1, \ldots, l) \qquad (s = 1, \ldots, p)$$

we obtain the equations of motion in a form which does not contain the reactions.

It is necessary to note that the reduction of the order of the system, using the equations of the holonomic constraints, can be carried out only after all the operations indicated on the left-hand sides of Equations (4.6) on the function T° depending on q_i°, $\dot{q}_i^\circ$, t, have been performed.

If, in the above-mentioned examples, as the Lagrangian coordinates are taken the Cartesian coordinates x, y of the particle in its plane of motion, the origin being at the point O, the x-axis directed horizontally to the right and the y-axis downward, then in the case of the first and the second example the axiom of reactions for N_x and N_y respectively is

$$N_x / x = N_y / y, \qquad N_x / y = - N_y / x$$

Equations (4.2) reduce to $x\delta x + y\delta y = 0$ in the first and to $x\delta y - y\delta x = 0$ in the second example. The integrals are $x^2 + y^2 = r^2$ and $x/y = u = \tan \alpha$, respectively.

The definition of virtual displacements can be taken in the case of the first example in the form $\delta r = 0$ and in the case of the second example in the form $\delta \alpha = 0$. No wonder, therefore, that if $T^\circ = 1/2(\dot{r}^2 + \dot{\alpha}^2 r^2)$, then Equations (1.1) and (1.2) will be

$$\frac{d}{dt}\frac{\partial T^\circ}{\partial \dot{\alpha}} - \frac{\partial T^\circ}{\partial \alpha} = Q_\alpha, \qquad \frac{d}{dt}\cdot\frac{\partial T^\circ}{\partial \dot{r}} - \frac{\partial T^\circ}{\partial r} = Q_r$$

Note. If the axiom of reactions is linear and does not contain the active forces, then by means of operation (4.2) it is possible to introduce a linear, and from the active forces Q_j, independent definition of "virtual displacements" [4] in such a way that this definition together with the principle of "virtual displacements" gives the equations of motion which are equivalent to (3.2). This can be done even if the coefficients b_{ij}° depend on the generalized velocities. Since any definition of "virtual displacements" which is linear and independent of the active forces leads to a linear axiom of reactions which is independent of the active forces, then the condition mentioned is also a necessary one. Such constraints must be denoted as constraints of the Béghin type.

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